Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 17, 2004*
Accepted: *November 5, 2004*

Universal correlators from geometry

Robbert Dijkgraaf,[ab] **Annamaria Sinkovics**[a] **and Mine Temürhan**[a]

[a] *Institute for Theoretical Physics, University of Amsterdam*
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
[b] *Korteweg-de Vries Institute for Mathematics, University of Amsterdam*
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
E-mail: rhd@science.uva.nl, sinkovic@science.uva.n, minet@science.uva.nl

Abstract: Matrix model correlators show universal behaviour at short distances. We provide a derivation for these universal correlators by inserting probe branes in the underlying effective geometry. We generalize these results to study correlators of branes and their universal behaviour in the Calabi-Yau crystals, where we find a role for a generalized brane insertion.

Keywords: Topological Strings, Matrix Models.

JHEP11(2004)012

http://jhep.sissa.it/archive/papers/jhep112004012/jhep112004012.pdf

Contents

1. **Introduction** 1

2. **Branes and the WKB approximation** 3

3. **Universal correlator in random matrix models** 6

4. **Universal correlator from geometry** 7

5. **Correlator from two brane insertions** 10

6. **Correlators in crystal melting** 11
   - 6.1 Partitions on the plane 12
   - 6.2 3D partitions 12
   - 6.3 Brane picture 14
   - 6.4 Universality for 3D partitions 17

A. **Universal kernel from Hermite polynomials** 18
   - A.1 Endpoint asymptotics 19

1. Introduction

Topological strings on Calabi-Yau manifolds provide diverse connections between string theory, geometry and random matrix models. In particular it has been shown that B-model topological strings on certain non-compact Calabi-Yau backgrounds can be described in terms of matrix models [1, 2, 3], where the geometry emerges from the planar limit of the random matrix integral. Here the matrix model appears as the open strings living on compact branes.

The geometry of these B-models can be effectively analyzed by making use of non-compact B-branes [4]. Since these branes have infinite world-volume they are infinitely heavy and should be considered as non-dynamical external probes. More specifically, we are considering non-compact Calabi-Yau manifolds that are given as the hypersurface

$$zw - H(y,x) = 0\,, \tag{1.1}$$

where $z, w, y, x \in \mathbf{C}$. The non-compact branes are then parametrized by a fixed point in the (x, y) plane that lies on the curve

$$\mathcal{C}:\ H(x,y) = 0\,,$$

and they extend in the coordinate z or w.

The B-model describes the complex structure deformations of the complex curve $\mathcal{C}$ or, equivalently, of the "hamiltonian" $H(x,y)$. The variations of the complex structure at infinity can be introduced by a chiral boson, $\phi(x)$. In a local coordinate patch this chiral boson is defined by

$$y(x) = \partial\phi(x)\,,$$

and it describes the variation of the curve through its parametrization $y(x)$.

The target space theory of the B-model on this geometry is the Kodaira-Spencer theory [5] where $\partial\phi$ is the dimensional reduction of the KS field A. The B-branes can be thought as defects for the KS field $\phi(x)$. The operator $\psi(x)$ that creates or annihilates a brane turns out to be a free fermion field and is related to the chiral boson via the familiar bosonization formula

$$\psi(x) = e^{i\phi(x)/g_s}\,.$$

Similarly the field $\psi^*(x) = e^{-i\phi/g_s}$ creates/annihilates an anti D-brane.

In terms of the matrix model, the chiral boson is the collective field of the eigenvalues of the matrix. The fermions are basically free fermions, but transform between the different patches as wavefunctions with generalized Fourier transformation. We will make use of this fact by choosing good coordinates, and later Fourier transforming back to the original coordinates.

We will use the fermionic formulation to study certain correlators in the matrix models. In random matrix models, it has been understood for a long time that eigenvalue correlators have an interesting behaviour at short scales, called "universality" [6]. In particular, the joint probablity of n eigenvalues is given by a determinant of a single kernel

$$\rho(\lambda_1,\ldots,\lambda_n) = \det_{n\times n} K(\lambda_i,\lambda_j)\,.$$

In the limit N large, while keeping the rescaled distances $N(\lambda_i-\lambda_j)$ fixed, the kernel K takes the form

$$K(\lambda_i,\lambda_j) \sim \frac{\sin N\pi\rho(\bar\lambda)(\lambda_i-\lambda_j)}{N\pi(\lambda_i-\lambda_j)}\,, \tag{1.2}$$

where $\rho(\bar\lambda)$ is the density of eigenvalues at the mean $\bar\lambda = (\lambda_i+\lambda_j)/2$. The formula is called universal because it does not depend explicitly on the form of the matrix model potential, thus it has the same form for any potential. It has only a functional dependence on the potential through the relatively uninteresting scaling factor of the mean density.

There are many ways to derive this kernel in random matrix models, the usual one relies on introducing orthogonal polynomials. With the use of brane insertions, we provide a simple derivation by writing the kernel in terms of the correlator of free fermions as

$$K(x_1,x_2) = \langle\psi(x_1)\psi^*(x_2)\rangle \sim \frac{e^{i(\phi(x_1)-\phi(x_2))/g_s}}{x_1-x_2}\,, \tag{1.3}$$

where the second equality is by bosonization. Here $\phi(x)$ is the chiral boson which is related to the geometry associated to the matrix model. In this formula, the use of good coordinates, which are single valued, is an essential point. That is, it only takes this simple form if the coordinate x parametrizes a unique point on the curve $\mathcal{C}$. It is only in these coordinates the branes, or free fermions, can be inserted at a definite position.

As we will explain, to find the kernel in the usual double valued coordinates, we can use Fourier transformation. In fact, we are just applying standard techniques of the semiclassical WKB approximation. The kernel will then be given as a weighted sum of contributions, coming from the "images" of the brane in the multivalued coordinates. In the multicover coordinates, the position of the brane cannot be fixed unambigously, and in fact, the brane insertion is best defined via the Fourier transformation. A version of this idea first appeared in [7] using methods of conformal field theory. Here we make it more explicit and reformulate it in the recent language of brane insertions.

The universal correlator is an interesting quantity to consider, since it probes the geometry at short distances. We can investigate it with the idea of inserting probe branes of the geometry. In fact, brane probes can also be used in the more general context of the Calabi-Yau crystal [8, 9, 10].

We study certain correlators in the crystal which are the 3 dimensional analogues of the eigenvalue correlators of matrix models. We find that such correlators can be described by fermion insertions in the geometry. These fermions are "generalized" in the sense they depend on two parameters. They are the usual one dimensional chiral fermions, but inserted at an arbitrary slice of the crystal, giving an additional parameter. Thus the fermions are effectively two dimensional objects probing the 3 dimensional structure of the crystal.

The generalized fermions do not correspond to the lagrangian brane probes in [10]. They are different objects probing the interior of the crystal. Finding the precise brane interpretation is an interesting open problem.

It is natural to ask whether the free fermion correlators of the crystal also show a similar universal behaviour when scaling parameters appropriately. In fact such correlators have been computed in the mathematics literature of random partitions [11], and a similar structure of universal correlators is found.

When taking a two dimensional slice of the crystal, in the scaling region we find the same sine kernel as in the case of random matrices. It is well-known in the mathematics of partitions that one can define a suitable probability measure which is the discretized version of the measure for gaussian matrix models. The three dimensional analogue of the sine kernel is a more complicated object, written in terms of an incomplete beta-function. We conjecture this beta-function is the universal scaling limit of a certain 2-matrix model with a unitary measure, which appears in the context of Chern-Simons theory [12], and the topological vertex [13].

2. Branes and the WKB approximation

To understand the relation between eigenvalues in matrix models and brane insertions in topological strings, let us first review some familiar facts about semi-classical quantization. We consider (x, y) as coordinates on a two-dimensional phase space. The curve $\mathcal{C}$ given by $H(x, y) = 0$ can be viewed as the orbit that corresponds to the classical ground state of the system described by the hamiltonian H. To such a one-dimensional curve one can

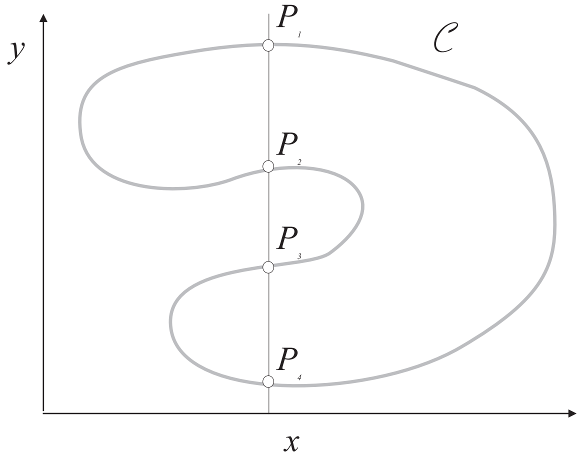

Figure 1: The graph of the function $y(x)$ is given by the orbit $\mathcal{C}$ determined by $H(x, y) = 0$. In general it has locally several branches: to a single value of x correspond the geometric points $P_1, P_2, \ldots$

associate the (global) action variable μ given by the contour integral

$$\mu = \frac{1}{2\pi} \oint_{\mathcal{C}} y dx \,.$$

The Bohr-Sommerfeld rule quantizes this as $\mu = (N + \frac{1}{2})g_s$, where we identify here and subsequently $\hbar$ with the string coupling constant g_s. It is useful to think of the hamiltonian $H(x, y)$, and therefore also the curve $\mathcal{C}$, as part of a family of systems parametrized by the modulus μ.

In general, if we want to use x as a coordinate and solve for y as a function of x on the curve $H(x, y) = 0$, there can be several branches $y_I(x)$. This is because the line of constant x will intersect the curve $\mathcal{C}$ in different points P_I, see figure 1.

All these points at these different sheets will contribute to the semi-classical wavefunction $\psi(x)$ associated to the orbit $\mathcal{C}$. The phase of this wavefunction is given by the local action

$$\phi_I(x) = \int_{x_0}^{x} y_I(x')dx'$$

with x_0 some arbitrary reference point. The full WKB expression for the wave function reads with some overall normalization c

$$\psi(x) = c \cdot \sum_I \left(\frac{\partial^2 \phi_I}{\partial x \partial \mu} \right)^{1/2} \exp\left[i \frac{\phi_I(x)}{g_s} \right].$$

Here all the intersection points P_I contribute. For example, for the harmonic oscillator with

$$H(x, y) = x^2 + y^2 - 4\mu$$

we have two such points. Indeed the wavefunction is well-known to be given in the WKB approximation by (for the "allowed zone")

$$\psi(x) \sim (4\mu - x^2)^{-\frac{1}{4}} \cos\left(\int^x \sqrt{4\mu - x'^2} dx' - \frac{\pi}{4} \right).$$

JHEP11(2004)012

In the context of matrix models for the topological B-model, the geometric points P_I on the curve $\mathcal{C}$ correspond to non-compact D-branes on the Calabi-Yau space. The variable x appears naturally in the matrix model as the eigenvalue, but will not be a good coordinate, since the spectral curve $\mathcal{C}$ will always be a multiple cover over the x-plane (a double cover for a single matrix integral). The natural fermion operators $\psi(x)$, that appear in the matrix model and correspond to creating a single eigenvalue, are therefore not represented by a single geometric brane. Instead they are a superposition of branes inserted at the inverse images P_I of x, just as we have in the formula for the WKB wavefunction.

In particular for the local geometry with two sheets, where we forget about all the x-dependence and therefore can write the local geometry as

$$y^2 = p^2\,,$$

we have branes inserted at the points P_1 and P_2 given by $y_{1,2} = \pm p$. The corresponding action functions are $\phi_1(x) = px$ and $\phi_2(x) = -px$. The wavefunction is naturally given as the sum

$$\psi(x) \sim p^{-1/2}\left[\exp i\left(\frac{px}{g_s}+\frac{\pi}{4}\right)+\exp i\left(-\frac{px}{g_s}-\frac{\pi}{4}\right)\right].$$

That is, we have in terms of brane insertions the natural identification

$$\psi(x) = e^{\frac{i\pi}{4}}\psi(P_1) + e^{-\frac{i\pi}{4}}\psi(P_2)\,.$$

Similarly the anti-brane is inserted by

$$\psi^*(x) = e^{\frac{i\pi}{4}}\psi^*(P_1) + e^{-\frac{i\pi}{4}}\psi^*(P_2)\,.$$

(Note that this is not the complex conjugated expression.)

With this translation we can easily compute multi-fermion correlation functions. Here we should remind us that we will have only contraction of operators that create or annihilate branes on the same sheet. For example, we have the two-point function

$$\big\langle\psi(x)\psi^*(x')\big\rangle = e^{\frac{i\pi}{2}}\left\langle\psi(P_1)\psi^*(P_1')\right\rangle - e^{-\frac{i\pi}{2}}\left\langle\psi(P_2)\psi^*(P_2')\right\rangle.$$

This can be evaluated in the limit $x \to x'$ as

$$\big\langle\psi(x)\psi^*(x')\big\rangle \sim \frac{\sin\left(\sqrt{\mu}(x-x')/g_s\right)}{\sqrt{\mu}(x-x')}$$

which gives the famous sine-kernel.

This description becomes problematic at caustics or turning points where the curve $\mathcal{C}$ is perpendicular to the x-direction and two of the branches coalesce as in figure 2. The right description at these points is to use the coordinate y instead. The new wavefunction is then determined by a Fourier transform

$$\psi(y) = \int dx\ e^{ixy/g_s}\psi(x)\,.$$

As we will see later this naturally leads to the Airy kernel.

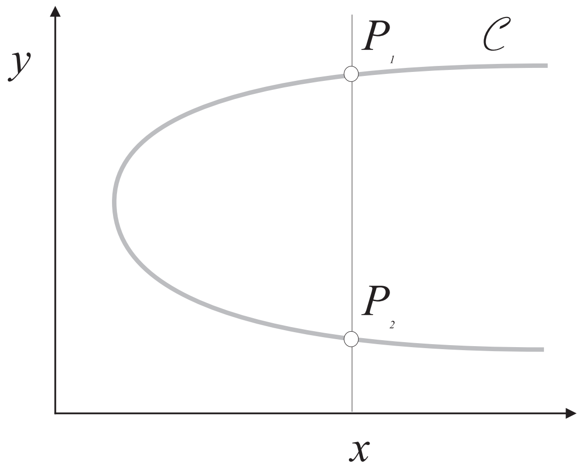

Figure 2: Close to a turning point, given by $y^2 = x$, the good coordinate is y not x.

3. Universal correlator in random matrix models

We now turn to a detailed investigation of the universal correlators in matrix models using the geometric picture of brane insertions just described. Consider an $N \times N$ matrix model with partition function

$$Z = \int d\Phi\, e^{-\frac{1}{g_s}\operatorname{Tr} W(\Phi)} \tag{3.1}$$

and let us define the density of n of its eigenvalues as

$$\rho(\lambda_1, \dots \lambda_n) = \frac{(N-n)!}{N!}\Big\langle \prod_{k=1}^{n} \operatorname{Tr} \delta(\lambda_k - \Phi)\Big\rangle\,. \tag{3.2}$$

For the random matrix model this describes the probability of measuring n eigenvalues at the same time. More precisely, the interesting part of this correlation function is its connected part. In the large-N limit the multicorrelator factorizes into one point functions, and therefore only its connected component contains new information.

Introducing orthogonal polynomials

$$\int d\lambda e^{-\frac{W(\lambda)}{g_s}} P_n(\lambda) P_m(\lambda) = \delta_{nm}$$

the joint eigenvalue distribution can be expressed [6, 14] as a determinant a single two-point kernel

$$\rho_n(\lambda_1, \dots, \lambda_n) = \frac{N^n (N-n)!}{N!} \det_{n\times n} K(\lambda_i, \lambda_j)\,. \tag{3.3}$$

In terms of the wavefunctions

$$\Psi_n(\lambda_i) = P_n(\lambda_i) e^{-\frac{W(\lambda)}{2g_s}}$$

this kernel can be written as

$$K(\lambda_i, \lambda_j) = \frac{1}{N}\sum_{n=0}^{N-1} \Psi_n(\lambda_i)\Psi_n(\lambda_j) = \frac{\Psi_N(\lambda_i)\Psi_{N-1}(\lambda_j) - \Psi_{N-1}(\lambda_i)\Psi_N(\lambda_j)}{\lambda_i - \lambda_j}\,. \tag{3.4}$$

Here we dropped the overall normalization factors, and at the second equality used the Darboux-Christoffel formula for the orthogonal polynomials. This formula can also be interpreted as a Slater determinant for second quantized fermions

$$K(\lambda_i, \lambda_j) = \langle \Psi^\dagger(\lambda_i)\Psi(\lambda_j)\rangle \,.$$

The eigenvalues of matrix model are usually located along a cut on the real axis. When scaling close to a fixed point in the middle of the cut, this kernel satisfies universal properties

$$K(\lambda_i, \lambda_j) \sim \frac{\sin\left[N\pi(\lambda_i - \lambda_j)\rho(\bar\lambda)\right]}{N\pi(\lambda_i - \lambda_j)} \tag{3.5}$$

in the region where $(\lambda_i - \lambda_j) \sim O(1/N)$. Here $\bar\lambda = (\lambda_i + \lambda_j)/2$, and $\rho(\bar\lambda)$ is the single eigenvalue density.

When scaling close to the endpoint of the cut, the kernel has a different expression in terms of Airy functions

$$K(\lambda_i, \lambda_j) \sim \frac{\mathrm{Ai}(\lambda_i)\mathrm{Ai}'(\lambda_j) - \mathrm{Ai}(\lambda_j)\mathrm{Ai}'(\lambda_i)}{\lambda_i - \lambda_j}\,, \tag{3.6}$$

where the $\lambda's$ are now suitably scaled variables close to the endpoint.

Since the form of the kernel does not depend on the specific form of the potential, we can restrict attention to the case of gaussian potential $V(x) = x^2/2$. For the gaussian potential, the sine and the Airy kernel can be derived by using the direct asymptotic expansion of the wavefunctions, which in this case are Hermite polynomials. We include this standard computation in the appendix.

4. Universal correlator from geometry

In the recent context of topological string theory, matrix models are directly related to the geometry of the CY. In the limit $N \to \infty$, $g_s \to 0$, $\mu = g_s N$ fixed, the CY geometry associated to the matrix model is

$$zw + y(x)^2 + (W'(x))^2 - f(x) = 0\,.$$

Here the quantum deformation $f(x)$ is a polynomial computed from the potential as

$$f(x) = \frac{4\mu}{N}\sum_i \frac{W'(x) - W'(\lambda_i)}{x - \lambda_i}\,,$$

and

$$y(x) = \partial\phi(x)$$

is the chiral boson from the collective field

$$\phi(x) = 2g_s\,\mathrm{Tr}\log(x - \Phi) - W'(x)\,.$$

The Riemann surface associated with the gaussian matrix model in this way is given by

$$H(y, x) = y^2 + x^2 - 4\mu = 0\,. \tag{4.1}$$

As discussed in [4], the geometry can be probed by inserting non-compact branes. Taking an asymptotic patch in the geometry, and considering the worldvolume action of the noncompact brane one finds x and y are a symplectic pair of variables with canonical commutation relations $[x, y] = i g_s$. In terms of operators,

$$y = -i g_s \frac{\partial}{\partial x}\,.$$

Inserting a non-compact brane in this geometry corresponds to adding a fermion $\psi(x) = e^{i\phi(x)/g_s}$ with wavefunction

$$\Psi(x) = \langle \psi(x) \rangle = e^{i\phi_{\rm cl}(x)/g_s}\,. \tag{4.2}$$

This wavefunction satisfies the Schrödinger equation

$$H\left(-i g_s \frac{\partial}{\partial x}, x\right) \Psi(x) = 0\,. \tag{4.3}$$

For the gaussian matrix model these wavefunctions are Hermite polynomials. In fact in the gaussian case these are the same wavefunctions as the orthogonal polynomial wavefunctions of the previous section. (In the general case the orthogonal polynomial wavefunctions do not satisfy the Schrödinger equation, due to more complicated normal ordering issues.)

In this case, we can simply derive the kernel by finding the wavefunctions from the geometry, and inserting in the Darboux-Christoffel formula. Let us first show this for the Airy kernel. When close to the branch points, the curve is simply

$$y^2 + x = 0\,, \tag{4.4}$$

where we rescaled x for convenience, and put the branch point at $x = 0$. We can insert a brane in a definite position in the y-coordinate which is single valued. In the y coordinate the wavefunction solves

$$\left(y^2 + i g_s \frac{\partial}{\partial y}\right) \Psi(y) = 0\,,$$

so it is given by

$$\Psi(y) = e^{i\frac{y^3}{3g_s}}\,.$$

(where we dropped the overall constant.) Transforming back to the x coordinates we get

$$\Psi(x) = \int dy\, e^{\frac{ixy}{g_s}} \Psi(y) = {g_s}^{1/3} \mathrm{Ai}\left[{g_s}^{-2/3} x\right].$$

In the large-N limit the Darboux-Christoffel formula becomes

$$K(x_1, x_2) \sim \frac{\Psi(x_1)\Psi'(x_2) - \Psi(x_2)\Psi'(x_1)}{x_1 - x_2}\,,$$

where $\Psi(x)$ are now the wavefunctions defined from the planar geometry, keeping N large and $\mu = g_s N$ fixed. So from the Darboux-Christoffel formula we immediately have

$$K(x_1, x_2) \sim \frac{\mathrm{Ai}(x_1)\mathrm{Ai}'(x_2) - \mathrm{Ai}(x_2)\mathrm{Ai}'(x_1)}{x_1 - x_2},$$

where $x_{1,2}$ are rescaled coordinates measuring distances from the branch points.

The sine kernel can be derived in a similar way by scaling to a fixed point in the cut away from the branch points. In this case the curve reduces to

$$y^2 + \bar{x}^2 - 4\mu + 2x\bar{x} = 0,$$

where $\bar{x}$ is the point we scale to, and x is the distance to that point. Let us work again in the y coordinates. If dropping the small x-term, the wavefunction solves

$$(y^2 - p(\bar{x})^2)\Psi(y) = 0,$$

where $p(\bar{x}) = \sqrt{4\mu - \bar{x}^2}$ is the classical momentum at $\bar{x}$. The wavefunction is a sum of δ-functions

$$\Psi(y) = e^{\frac{i\pi}{4}}\delta(y - p) + e^{-\frac{i\pi}{4}}\delta(y + p).$$

The phases are fixed using the WKB approximation discussed in section 2. Fourier transforming back

$$\Psi(x) = e^{\frac{i\pi}{4}} e^{\frac{ipx}{g_s}} + e^{-\frac{i\pi}{4}} e^{-\frac{ipx}{g_s}}.$$

Choosing the symmetric combination and substituting back in the Slater determinant formula we get the sine kernel.

A more precise way is to keep the small x-term,

$$\left(y^2 - p(\bar{x})^2 + 2ig_s\bar{x}\frac{\partial}{\partial y}\right)\Psi(y) = 0.$$

The corrected solution is

$$\Psi(y) = \exp\left[\frac{i}{2g_s\bar{x}}\left(\frac{y^3}{3} - p(\bar{x})^2 y\right)\right].$$

Transforming back to x the solution is again Airy function

$$\Psi(x) = (2g_s\bar{x})^{1/3}\mathrm{Ai}\left[\left(\frac{2\bar{x}}{g_s^2}\right)^{1/3}\left(x - \frac{p(\bar{x})^2}{2\bar{x}}\right)\right]. \tag{4.5}$$

Now we still have to expand in x, which is the small distance to the fixed point $\bar{x}$. Doing a saddle point expansion of the Airy integral we pick up the contribution from the two saddle points

$$y_{1,2} = \pm\sqrt{p^2 - 2x\bar{x}}.$$

Substituting in the Darboux-Christoffel formula

$$K(x_1, x_2) \sim \frac{\sin p(x_1 - x_2)}{x_1 - x_2}$$

we again get the sine kernel.

5. Correlator from two brane insertions

An even shorter way to derive the universal correlators is to insert two branes in the geometry. In the single valued coordinates, where the branes can be inserted in definite position, the kernel can be expressed as [7]

$$K(y_1, y_2) = \langle \psi(y_1)\psi^*(y_2)\rangle \,. \tag{5.1}$$

An intuitive derivation of this formula would express the joint correlators in the collective field, and then rewrite it with bosonization. However, the x-coordinates here are really inconvenient, because they are double valued, and an inconvenient averaging is involved. This averaging was first described in [7], and can be understood as summing over the images of the brane in the multicover coordinate, as discussed in detail in section 2.

In the single-valued y-coordinate, we start with the formula (5.1), which can now be interpreted as an insertion of a brane and an antibrane in the geometry. The branes inserted are free fermions, consisting of a classical wavefunction, and a quantum part

$$\psi(y) = \Psi(y)\psi_{\rm qu}(y) = e^{\frac{i\phi_{\rm cl}(y)}{g_s}}\psi_{\rm qu}(y)\,.$$

So the short-distance correlator in the y-coordinates is

$$\langle \psi(y_1)\psi^*(y_2)\rangle = \frac{e^{\frac{i}{g_s}(\phi(y_1)-\phi(y_2))}}{y_1-y_2}\,.$$

Consider now again the geometry close to the branch point,

$$y^2 + x = 0\,.$$

The classical wavefunction from the geometry is

$$\Psi(y) = e^{\frac{i}{g_s}\phi(y)} = e^{\frac{i}{g_s}y^3/3}\,.$$

Transforming back to the x-coordinates

$$\langle \psi(x_1)\psi^*(x_2)\rangle = \int dy_1 dy_2\ \frac{\exp \frac{i}{g_s}\left(x_1y_1 - x_2y_2 + y_1^3/3 - y_2^3/3\right)}{y_1-y_2}\,.$$

The Fourier transform is easily evaluated by noting that

$$e^{ix_1y_1 - ix_2y_2} = \frac{-i}{x_1-x_2}(\,\partial_{y_1}+\partial_{y_2})\,e^{ix_1y_1-ix_2y_2}\,.$$

Substituting and after a partial integration in rescaled coordinates we arrive at

$$\langle \psi(x_1)\psi^*(x_2)\rangle \sim \frac{\mathrm{Ai}(x_1)\mathrm{Ai}'(x_2) - \mathrm{Ai}(\mathrm{x}_2)\mathrm{Ai}'(\mathrm{x}_1)}{x_1-x_2}\,.$$

Let us expand this formula in the limit where, while close to the endpoint, the distance between the two points is very small, so $x_1 = \bar{x}+x$, $x_2 = \bar{x}-x$, where x is the small distance.

Doing a saddle point expansion of the Airy integral we pick up the contribution of the two saddle points $y = \pm\sqrt{\bar{x}}$. Substituting we arrive at the sine kernel formula

$$\langle \psi(x_1)\psi^*(x_2)\rangle \sim \frac{\sin x\sqrt{\bar{x}}}{x}\,.$$

This is consistent with the previous expression for the sine kernel, since close to the endpoint $y^2+x=0$ the density scales as $\rho(\bar{x}) \sim \sqrt{\bar{x}}$. Doing the saddle point expansion of the Fourier transform, and picking up the contributions from the two saddle points, we automatically introduced an averaging in the double valued x-coordinates.

We could also consider the more general geometry

$$y^m + x = 0\,.$$

In this case the wavefunctions in the y-patch are

$$\Psi(y) = \exp\left(\frac{iy^{m+1}}{g_s(m+1)}\right).$$

Fourier transforming in the x-patch this gives

$$\Psi(x) = \int dy \exp\frac{1}{g_s}\left(ixy + i\frac{y^{m+1}}{(m+1)}\right).$$

The two point correlator and its scaling limits can be similarly derived.

6. Correlators in crystal melting

Recently a connection has been discovered between the statistical mechanics picture of crystal melting and A-model topological strings on Calabi-Yau manifolds. The crystal describes the toric base of the Calabi-Yau, with lattice spacing of order g_s. The temperature of the melting crystal is $1/g_s$ [8, 9]. At very large scales the Calabi-Yau is described in terms of classical geometry. Decreasing the distance scale to the string scales, the geometry takes the form of a smooth limit shape. At even smaller distances a "gravitational foam" picture emerges.

The structure of the crystal can be probed by putting probe branes in the geometry. The effect of adding a single brane probe was first studied in [9], where the correction to the partition function was discussed.

Addition of multiple branes and antibranes was considered recently in [10]. In this paper, a statistical interpretation of non-compact probe branes as defects in the Calabi-Yau crystal is given, and correlators for inserting branes are derived. These results are in accordance to that non-compact brane probes can be thought of as adding free fermions with a certain geometric transformation law [4], at the B-model side.

Similarly as for the random matrices, we will investigate certain correlators by inserting branes in the geometry. The correlators we are interested in are the 3D crystal analogues of the eigenvalue correlators of random matrices.

JHEP11(2004)012

6.1 Partitions on the plane

Let us first consider partitions on the plane. If we draw the diagram of a partition of N, and rescale it with $\sqrt{N}$, a smooth limit curve of the diagram emerges [11, 15]. This is the two-dimensional analogue of the limit shape of the melting crystal. In general, the boundary is a graph of a piecewise linear function, consisting from jumps ups and downs. Correlators in this context measure the probability of a given pattern of up and down jumps. In fact it is enough to consider the probablities to measure the set of down jumps, since this already gives the complete information about the shape of the partition. When scaling N large, so that the elements in the set of downs x_i

$$\frac{x_i}{\sqrt{N}} \in [-2, 2]$$

such probabilities are given by the determinant of a single discrete sine kernel [11, 15]

$$K_{\sin}(x_i, x_j) = \frac{\sin \phi_i (x_i - x_j)}{\pi(x_i - x_j)} \qquad \phi_i = \cos^{-1}\left(\frac{x_i}{2\sqrt{N}}\right). \tag{6.1}$$

Thus the same sine kernel as for random matrices emerges. The important difference however is that in this case x_i are discrete, while in the random matrix case we had a continuous distribution. Also the scaling factor ϕ is different from the random matrix case where we had a scaling with the density, but such scalings can always be absorbed in the redefinition of the variables.

The endpoint formula for the random matrix correlator in terms of Airy function also has a counterpart in random partitions.

6.2 3D partitions

We will now study similar correlators in 3D partitions. In the 3D generalization we can build a partition from a sequence of diagonal slices, which satisfy the interlacing condition, [9, 8, 11].

Another way of looking at the partition is viewing it from the top, and projecting it to a two-dimensional tiling pattern.[1] If the partition is located in the positive corner of (x, y, z) plane, the position of plaquettes in the tiling pattern is given in the discrete coordinates

$$\begin{aligned} t &= y - x\,, \\ h &= z - \frac{1}{2}(y + x)\,. \end{aligned}$$

One can then ask for the probability of measuring a set of fixed plaquettes, at positions $\{(t_1, h_1) \dots (t_n, h_n)\}$. These correlators are given as a determinant of a 3D kernel [16]

$$\begin{aligned} &K_{3D}((t_i, h_i), (t_j, h_j)) = \\ &\quad = \frac{1}{(2\pi i)^2} \int_{|s|=1\pm\epsilon} \int_{|w|=1\mp\epsilon} ds dw \; \frac{\Phi_{3D}(t_1, s)\Phi_{3D}^{-1}(t_2, w)}{s - w} \frac{1}{s^{h_i + \frac{|t_i|}{2} + \frac{1}{2}} w^{-h_j - \frac{|t_j|}{2} + \frac{1}{2}}}\,. \end{aligned} \tag{6.2}$$

[1] The projection is best understood in terms of pictures, see for example [11, 15].

Here the contour integral the top/bottom $\pm$ signs are valid when $t_1 \geq t_2$ and and $t_1 < t_2$ respectively, $\Phi_{3D}^{-1} = 1/\Phi_{3D}$, and

$$\Phi_{3D}(s,t) = \frac{\prod_{m>max(0,-t)} (1-q^m/s)}{\prod_{m>max(0,t)} (1-q^m s)}, \qquad m \in \mathbf{Z} + \frac{1}{2}. \tag{6.3}$$

This kernel measures the probability of finding two fixed plaquettes in the random tiling pattern. In fact, this two-point correlator can be thought of as a non-compact brane and anti-brane probe insertion in the geometry.

The A-model geometry is related by mirror symmetry to B-model on the mirror Calabi-Yau [4]

$$zw - e^{-u} - e^{v} + 1 = 0 .$$

Inserting a single brane, the wavefunction is an eigenfunction of the hamiltonian

$$\begin{aligned} H(u,v)\, L(u,q) &= 0 , \\ H(u,v) &= q^{-1/2} e^{-u} + e^{-g_s \partial_u} - 1 , \qquad q = e^{-g_s} . \end{aligned} \tag{6.4}$$

The one-point function which satisfies this equation is the quantum dilogarithm

$$L(u,q) = \prod_{n=0}^{\infty} (1 - e^{-u} q^{n+\frac{1}{2}}) . \tag{6.5}$$

Similarly, the one point function for an antibrane is $L^*(u) = 1/L(u)$.

Let us introduce the new variables

$$\begin{aligned} s &= e^{u} , \\ w &= e^{\tilde{u}} , \\ \tilde{h}_i &= h_i - \frac{1}{2} + \frac{|t_i|}{2} , \\ \tilde{\tilde{h}}_j &= h_j + \frac{1}{2} + \frac{|t_j|}{2} . \end{aligned}$$

In terms of these variables, the kernel can be rewritten as

$$K_{3D}(\tilde{h}_i, \tilde{\tilde{h}}_j, t_i, t_j) = \frac{1}{(2\pi i)^2} \int du d\tilde{u}\, \frac{1}{e^{u} - e^{\tilde{u}}} e^{-u\tilde{h}_i}\, e^{\tilde{u}\tilde{\tilde{h}}_j}\, \Phi_{3D}(u,t_i)\, \Phi_{3D}^{-1}(\tilde{u},t_j) . \tag{6.6}$$

This is essentially the same form as the two-point free-fermion function of a brane and antibrane considered in [4], now Fourier-transformed in the variables $(\tilde{h}_i, \tilde{\tilde{h}}_j)$. Here Φ_{3D} and Φ_{3D}^{-1} are the brane and antibrane one-point functions,

$$\begin{aligned} \Phi_{3D}(u,t) &= \begin{cases} \dfrac{L(u,q)}{L(-u+tg_s,q)} & t \geq 0 , \\ \dfrac{L(u-tg_s,q)}{L(-u,q)} & t < 0 \end{cases} \\ \Phi_{3D}^{-1}(\tilde{u},t) &= \frac{1}{\Phi_{3D}(u,t)} . \end{aligned}$$

The formula shows that shifting the coordinate t shifts us to another slice of the crystal, parametrized by t. In particular the wavefunction at the slice $t = 0$ is

$$\Phi_{3D}(u,0) = \frac{L(u,q)}{L(-u,q)} .$$

6.3 Brane picture

It is a natural question what is the precise brane configuration which reproduce the 3D correlators, and how these branes are inserted in the geometry. To understand this, we consider the picture of [10] where lagrangian branes were described as defects in the crystal.

Consider now the 3D crystal constructed from a sequence of interlacing 2d partitions, $\mu(t)$.[2] A single 2d partition is a nonincreasing sequence of non-negative integers $\mu = \{\mu_1, \mu_2, \ldots\}$. To build a 3D partition, such sequences are labeled by an integer t, and must satisfy the interlacing condition

$$\begin{aligned} \mu(t) &< \mu(t+1) \quad t < 0\,, \\ \mu(t+1) &< \mu(t) \quad t \geq 0\,, \end{aligned}$$

where $\mu(t+1) > \mu(t)$, if

$$\mu_1(t+1) \geq \mu_1(t) \geq \mu_2(t+1) \geq \mu_2(t) \cdots .$$

A single two-dimensional slice can be described as a state in the fermionic Fock-space. By introducing the set of down jumps of the tableaux and its transpose

$$a_i = \mu_i - i + \frac{1}{2}\,, \tag{6.7}$$

$$b_i = \mu_i^T - i + \frac{1}{2}\,, \tag{6.8}$$

the fermionic state describing the tableaux consisting of d boxes is given by

$$|\mu\rangle = \prod_{i=1}^{d} \psi^*_{-a_i} \psi_{-b_i}\,.$$

Using bosonization, we introduce

$$q^{L_0}|\mu\rangle = q^{|\mu|}|\mu\rangle\,,$$

where $|\mu|$ denotes the number of boxes, and the creation/annihilation operators

$$\Gamma_\pm(z) = e^{\phi_\pm(z)}\,.$$

Here $\phi_\pm(z)$ are the positive and negative mode part of the chiral boson $\phi(z)$, which is related to the complex fermion by bosonization

$$\psi(z) =: e^\phi(z) : \,.$$

In this language, introducing a D-brane corresponds to the insertion of a fermion. Discarding the zero mode part, the D-brane operator can be written as

$$\Psi_D(z) = \Gamma_-^{-1}(z)\Gamma_+(z)\,.$$

We use here and throughout the paper the standard framing $p = 0$.

[2]Here we follow the notation of [8] and [10].

The partition function of the crystal with the operators can be written as

$$Z(q) = \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-\frac{1}{2}}) \prod_{m=1}^{\infty} \Gamma_-(q^{-m+\frac{1}{2}})|0\rangle \,, \tag{6.9}$$

where $q = e^{-g_s}$. With the commutation relation

$$\Gamma_+(z)\,\Gamma_-(z') = (1 - z/z')^{-1}\Gamma_-(z')\,\Gamma_+(z) \,, \tag{6.10}$$

it is straightforward to show that the partition function is the McMahon function

$$Z(q) = \prod_{n=1}^{\infty}(1-q^n)^{-n} = M(q) \,.$$

In [10], lagrangian D-brane insertions were considered. These branes end on the axis, and have the geometry

$$y = x + a = z + a \,, \qquad a > 0 \,,$$

where we chose the brane to end at $y = a$. In [10] it was shown that introducing such a lagrangian brane at a position $a = g_s(N_0 + \frac{1}{2})$, corresponds to inserting a fermion D-brane operator $\Psi_D(q^{-(N_0+\frac{1}{2})})$ at the slice $t = N_0 + 1$,

$$\begin{aligned} Z_D(q, N_0) \;=\; & \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-\frac{1}{2}}) \times \\ & \times \prod_{m=1}^{N_0+1} \Gamma_-(q^{-m+\frac{1}{2}}) \Psi_D(q^{-(N_0+\frac{1}{2})}) \prod_{m=N_0+2}^{\infty} \Gamma_-(q^{-m+\frac{1}{2}})\,|0\rangle \,. \end{aligned}$$

Commuting through the operators gives the brane one-point function

$$\begin{aligned} Z_D(q, N_0) \;&=\; M(q)\,\xi(q) \prod_{n=1}^{\infty}(1 - e^{-g_s(N_0+\frac{1}{2})}q^{n-\frac{1}{2}}) \\ &=\; M(q)\xi(q)L(g_s(N_0 + \frac{1}{2}), q) \,, \end{aligned}$$

where we have expressed the wavefunction in terms of the overall McMahon function $M(q)$ and the quantum dilogarithm $L(u, q)$ we introduced before, and[3]

$$\xi(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \,.$$

These brane insertions depend on the single parameter N_0, expressing the fact that the branes correspond to the lagrangian geometry. Comparing with the brane insertions for the 3D correlators in the previous section, we see that our brane insertions are of more general kind, since they depend on the two independent parameters t and u. In fact, they correspond to the insertion of a D-brane operator $\Psi_D(z)$ at an arbitrary slice t.

[3] $\xi(q)$ is a normalization factor with respect to the string answer, so that $Z_{\text{crystal}} = \xi(q) Z_{\text{string}}$ [10].

Let us insert a fermionic operator $\Psi_D(q^{-(N+\frac{1}{2})})$ at the slice $t = N_0 + 1$. Note that N and N_0 are now independent. This is given by the operator expression

$$\begin{aligned} Z_D(q,N,N_0) &= \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-\frac{1}{2}}) \times \\ &\quad \times \prod_{m=1}^{N_0+1} \Gamma_-(q^{-m+\frac{1}{2}}) \Psi_D(q^{-(N+\frac{1}{2})}) \prod_{m=N_0+2}^{\infty} \Gamma_-(q^{-m+\frac{1}{2}}) \, |0\rangle \,. \end{aligned}$$

Substituting $\Psi_D(z) = \Gamma_-^{-1}(z)\Gamma_+(z)$ we obtain

$$\begin{aligned} Z_D(q,N,N_0) &= \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-\frac{1}{2}}) \times \\ &\quad \times \prod_{m=1}^{N_0+1} \Gamma_-(q^{-m+\frac{1}{2}}) \Gamma_-^{-1}(q^{-(N+\frac{1}{2})}) \Gamma_+(q^{-(N+\frac{1}{2})}) \prod_{m=N_0+2}^{\infty} \Gamma_-(q^{-m+\frac{1}{2}}) \, |0\rangle \,. \end{aligned}$$

Commuting first the $\Gamma_+(q^{-(N+\frac{1}{2})})$ to the right gives a factor

$$\prod_{m=N_0+2}^{\infty} (1 - q^{m-N-1})^{-1} = \frac{1}{L(-g_s(N+\frac{1}{2}) + (N_0+1)g_s, q)} \,.$$

The leftover $\Gamma_-^{-1}(q^{-(N+\frac{1}{2})})$ cancels then the Nth Γ_- in the sequence. The remainder is then the McMahon function, up to an extra factor making up for the missing Γ_- at the place N. This extra factor is

$$\prod_{n=1}^{\infty} (1 - q^{n+N}) = L\left(g_s\left(N + \frac{1}{2}\right), q\right) .$$

Thus

$$Z_D(q,N,N_0) = M(q) \, \frac{L(g_s(N+\frac{1}{2}), q)}{L(-g_s(N+\frac{1}{2}) + (N_0+1)g_s, q)} \,. \tag{6.11}$$

Comparing with the wavefunctions appearing in the 3D correlators we obtain

$$Z_D(q,N,N_0) = M(q) \Phi_{3D}\left(g_s\left(N + \frac{1}{2}\right), N_0 + 1\right) . \tag{6.12}$$

Since we have started with a fermionic insertion at $u = g_s(N + 1/2)$ and at the slice $t = N_0 + 1$, there is a precise agreement. The branes inserted in the 3D correlator with wavefunction $\Phi_{3D}(u,t)$ correspond to the fermionic insertions $\Psi(e^{g_s u})$ at the slice t. Note that we have the choice to insert the fermion at a slice $t \geq 0$, or at $t < 0$. Inserting at $t < 0$ precisely reproduces $\Phi_{3D}(u,t)$ for $t < 0$. The branes are generalized branes, in the sense that they depend on the two parameters u and t.

It is interesting to note that the Fourier transform of the wavefunction of the generalized branes $\Phi_{3D}(u,t)$ is the Wigner function. Let us take $t > 0$. Using that the generalized brane wavefunction is a product of a brane and a shifted antibrane wavefunction, the Fourier transform gives

$$W(g_s t, y) = \int du L(u, q) L^*(-u + g_s t, q) e^{-iuy} ,$$

where we recognize the Wigner function. The Wigner function appeared in the recent black hole interpretation of the topological string partition function [17]. It would be interesting to investigate this connection further.

6.4 Universality for 3D partitions

It is interesting to observe that a universal scaling behaviour also appears in the case of the correlators of the 3D random partitions. In the same scaling limit as in the two-dimensional case, the 3D kernel reduces to the incomplete β-kernel [11]

$$K_\beta(t, h) = \frac{1}{2\pi i} \int_{\bar{\eta}}^{\eta} dz (1 - z)^t \frac{1}{z^{h+t/2+1}} .$$

where the distances kept fixed are $t = t_i - t_j$ and $h = h_i - h_j$, and η is a density scaling parameter. This reduces to the plane sine-kernel on two different slices. If we take $t = 0$, or $x = y$ in the original coordinates

$$K_\beta(h) = \frac{\sin h\phi}{\pi h} ,$$

we obtain the sine-kernel in h (with $\eta = e^{i\phi}$). Taking $h + t/2 + 1 = 0$, or $x - z = 1$, and changing variables $z \to (1 - z)$

$$K_\beta(t) = \frac{\sin \tilde{\phi}(t + 1)}{\pi(t + 1)} ,$$

the kernel reduces to a sine kernel in $t' = t + 1$. Here $\tilde{\phi}$ is a redefined scaling variable. This means the kernel is likely connected to a two-matrix model. Taking a gaussian two-matrix model with

$$\int dX dY \, e^{-\frac{1}{g_s}(\frac{1}{2}X^2 + \frac{1}{2}Y^2 + V(X,Y))} ,$$

the universal kernel has the form of $K(x_i - x_j, y_i - y_j) = K(x, y)$. Reducing to $y = 0$ or $x = 0$ we have

$$\begin{aligned} K(x, 0) &\sim \frac{\sin \pi\rho x}{\pi x} , \\ K(0, y) &\sim \frac{\sin \pi\rho y}{\pi y} \end{aligned}$$

Thus reducing to the two slices would correspond to restricting to the eigenvalues of only X or only Y in the two-matrix model. It is clear that the corresponding two-matrix model must have an interaction term, otherwise the full correlator would be just a product of the two single-matrix (or single-slice) correlators. This is apparently not the case for the β-kernel.

The two-matrix model in question is most likely the one related to the Chern-Simons partition function [12] and the topological vertex [13].

To understand the full kernel, we need the four-point function in this two-matrix model, and its scaling limit. Such kernels has been considered in the context of Brownian motion [18] for the usual two-matrix models. Using the context of Brownian motion [19] could lead to a more precise identification.

The universal scaling formula (incomplete beta kernel) is a probe of the crystal at short distances. Translated to topological strings, it may indicate an interesting short distance scaling regime of string amplitudes.

Acknowledgments

We would like to thank M. Mariño, E. Verlinde and C. Vafa for discussions. This research is supported by NWO, the FOM Programme *String Theory and Quantum Gravity*, and the CMPA grant of the University of Amsterdam.

A. Universal kernel from Hermite polynomials

The universal kernel and its endpoint form can also be computed using orthogonal polynomials

$$\int d\lambda \, e^{-\frac{V(\lambda)}{g_s}} P_n(\lambda) P_m(\lambda) = \delta_{nm} \, .$$

Defining the wavefunctions

$$\Psi_n(\lambda_i) = P_n(\lambda_i) \; e^{-\frac{V(\lambda)}{2g_s}} \, ,$$

and using the Darboux-Christoffel formula, the kernel (up to a constant) can be expressed as

$$K(\lambda_i, \lambda_j) = \frac{1}{N} \sum_{n=0}^{N-1} \Psi_n(\lambda_i) \Psi_n(\lambda_j) = \frac{\Psi_N(\lambda_1)\Psi_{N-1}(\lambda_2) - \Psi_{N-1}(\lambda_1)\Psi_N(\lambda_2)}{\lambda_1 - \lambda_2} \, . \tag{A.1}$$

For the gaussian potential we have

$$\Psi_n(x) = \frac{1}{(2g_s\pi)^{\frac{1}{4}} 2^{\frac{n}{2}} \sqrt{n!}} \, H_n\left(\frac{x}{\sqrt{2g_s}}\right) e^{-\frac{x^2}{4g_s}}$$

where H_n are the Hermite polynomials. These are the wavefunctions of the harmonic oscillator with $\hbar = 2g_s$. Using $H_n'(x) = 2nH_{n-1}(x)$ we can rewrite (A.1) as

$$\kappa(\lambda_i, \lambda_j) = \frac{A_N A_{N-1}}{2N} e^{-\frac{\lambda_1^2}{4g_s} - \frac{\lambda_2^2}{4g_s}} \left(\frac{H_N(\frac{\lambda_1}{\sqrt{2g_s}}) H_N'(\frac{\lambda_2}{\sqrt{2g_s}}) - H_N(\frac{\lambda_2}{\sqrt{2g_s}}) H_N'(\frac{\lambda_1}{\sqrt{2g_s}})}{\lambda_1 - \lambda_2} \right) \tag{A.2}$$

where A_n is the normalization factor of the gaussian wavefunctions

$$A_n = \frac{1}{(2g_s\pi)^{\frac{1}{4}} 2^{\frac{n}{2}} \sqrt{n!}} \, .$$

We need $\kappa(\lambda_i, \lambda_j)$ in the limit $N \to \infty$, $g_s \to 0$, $\mu = g_s N$ fixed and large. To find the asymptotic expansion of the Hermite polynomials, it is useful to rewrite them in terms of parabolic cylinder functions $U(a, x)$ as

$$H_n(x) = 2^{n/2}\, e^{\frac{1}{2}x^2} \;, U\left(-n - \frac{1}{2}, \sqrt{2}x\right).$$

The asymptotic expansion of the parabolic cylinder function $U(a, x)$ is given in terms of the quantity $Y = \sqrt{4|a| - x^2}$. Since $a = -N - 1/2$, we expand in the region where a is large, negative. In this region the argument $\lambda_i/\sqrt{g_s} \sim \sqrt{N/\mu}\lambda_i \sim \sqrt{N}$, so x is moderately large. Using the asymptotic expansion in this regime, and neglecting lower order terms in the $1/N$ expansion, we find

$$U(-N - 1/2, x) = \frac{2\sqrt{N!}}{(2\pi)^{1/4}} \frac{1}{\sqrt{Y(x)}} \cos\left(\frac{1}{2}\int_0^x Y(x')dx' - \frac{N\pi}{2}\right).$$

Substituting in (A.2), and expanding when $\lambda_i - \lambda_j \sim O(1/N)$ we arrive at

$$K(\lambda_i, \lambda_j) \sim \frac{\sin N\pi\rho(\bar{\lambda})(\lambda_i - \lambda_j)}{\pi(\lambda_i - \lambda_j)},$$

where $\bar{\lambda} = (\lambda_i + \lambda_j)/2$, and $\rho(\bar{\lambda})$ is the normalized eigenvalue density

$$\rho(\bar{\lambda}) = \frac{1}{2\mu\pi}\sqrt{4\mu - \bar{\lambda}}\,.$$

The overall normalization factor can be fixed from the definitions, using that $\kappa(\lambda, \lambda) = \rho(\lambda)$. Finally

$$\kappa(\lambda_i, \lambda_j) = \frac{\sin N\pi\rho(\bar{\lambda})(\lambda_i - \lambda_j)}{N\pi(\lambda_i - \lambda_j)},$$

in agreement with (3.5).

A.1 Endpoint asymptotics

When scaling close to one of the branch points, $\lambda \to \pm 2\sqrt{\mu}$, the kernel has a different asymptotic expansion in terms of Airy functions. Let us scale to the positive branch point $2\sqrt{\mu}$. At large a the parabolic cylinder functions can be expressed in terms of Airy functions as

$$U(a, x) \sim 2^{-\frac{1}{4}-\frac{1}{2}a}\Gamma\left(\frac{1}{4} - \frac{1}{2}a\right)\left(\frac{t}{\xi^2 - 1}\right)^{\frac{1}{4}} \mathrm{Ai}(t)\,,$$

where

$$x = 2\sqrt{|a|}\xi\,, \qquad t = (4|a|)^{2/3}\tau\,,$$

and

$$\tau = \left(\frac{3}{8}(\xi\sqrt{\xi^2 - 1} - \cosh^{-1}\xi)\right)^{\frac{2}{3}}.$$

We need the scaling limit when $x \to 2\sqrt{|a|}$, that is when $\xi \to 1$. In this limit

$$U(a, x) \sim 2^{-\frac{1}{4}-\frac{1}{2}a}\Gamma\left(\frac{1}{4} - \frac{1}{2}a\right)|a|^{\frac{1}{6}}\,\mathrm{Ai(t)}\,,$$

where $t = 2|a|^{\frac{2}{3}}(\xi - 1)$ measures the distance from the branch point. Close to the branch point then the Hermite polynomials have the asymptotics

$$H_N(x) \sim 2^{\frac{N}{2}} e^{\frac{x^2}{2}} \left(N + \frac{1}{2}\right)^{\frac{1}{6}} \Gamma\left(\frac{1+N}{2}\right) \mathrm{Ai}(u)\,,$$

where

$$u = \left(N + \frac{1}{2}\right)^{\frac{1}{6}} \left(x - \sqrt{4N+2}\right).$$

Substituting in (A.2) we find

$$K(u_1, u_2) \sim \frac{\mathrm{Ai(u_1)Ai'(u_2) - Ai(u_2)Ai'(u_1)}}{u_1 - u_2}\,.$$

(Here we rescaled u by a $1/\sqrt{g_s}$ as doing the substitution.)

References

[1] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[2] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106].

[3] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[4] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, *Topological strings and integrable hierarchies*, hep-th/0312085.

[5] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[6] E. Brezin and A. Zee, *Universality of the correlations between eigenvalues of large random matrices*, *Nucl. Phys.* **B 402** (1993) 613.

[7] I.K. Kostov, *Conformal field theory techniques in random matrix models*, hep-th/9907060.

[8] A. Okounkov, N. Reshetikhin and C. Vafa, *Quantum Calabi-Yau and classical crystals*, hep-th/0309208.

[9] A. Iqbal, N. Nekrasov, A. Okounkov and C. Vafa, *Quantum foam and topological strings*, hep-th/0312022.

[10] N. Saulina and C. Vafa, *D-branes as defects in the Calabi-Yau crystal*, hep-th/0404246.

[11] A. Okounkov, *Symmetric functions and random partitions*, math.CO/0309074.

[12] M. Marino, *Chern-Simons theory, matrix integrals and perturbative three-manifold invariants*, hep-th/0207096.

[13] M. Aganagic, A. Klemm, M. Marino and C. Vafa, *Matrix model as a mirror of Chern-Simons theory*, *J. High Energy Phys.* **02** (2004) 010 [hep-th/0211098].

JHEP11(2004)012

[14] B. Eynard, *Eigenvalue distribution of large random matrices, from one matrix to several coupled matrices*, cond-mat/9707005.

[15] A. Okounkov, *Infinite wedge and random partitions*, math.RT/9907127.

[16] A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, math.CO/0107056.

[17] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, hep-th/0405146.

[18] C.A. Tracy and H. Widom, *Differential equations of Dyson processes*, math.PR/0309082.

[19] S. de Haro and M. Tierz, *Brownian motion, Chern-Simons theory and 2D Yang-Mills*, *Phys. Lett.* **B 601** (2004) 201 [hep-th/0406093].